JOURNAL OF DIFFERENTIAL EQUATIONS 42, 438–446 (1981)

Weak Solutions $u(x, t)$ to Parabolic Partial Differential Equations with Coefficients That Depend upon $u(y_l, \psi_l(t, u(x, t)))$, $l = 1, \ldots, k$

JOHN R. CANNON*

University of Texas, Austin, Texas 78712

AND

PAUL C. DUCHATEAU†

Colorado State University, Fort Collins, Colorado 80523

Received February 6, 1981

The existence of weak solutions $u(x, t)$ to parabolic partial differential equations with coefficients that depend on $u(y_l, \psi_l(t, u(x, t)))$, $l = 1, \ldots, k$, is demonstrated using a retardation of the time arguments in the coefficients along with regularity and compactness results for solutions of linear parabolic partial differential equations.

1. INTRODUCTION

Let Ω denote a domain in R^n with smooth boundary $\partial\Omega$. Let $Q_T = \Omega \times (0, T]$ and $S_T = \partial\Omega \times (0, T]$. Now consider the problem of finding an unknown function $u(x, t)$ satisfying

$$\begin{aligned} Lu(x, t) &= f(x, t) && \text{in } Q_T, \\ u(x, t) &= \phi(x), && t = 0, \quad x \in \Omega \\ u(x, t) &= 0, && (x, t) \in S_T, \end{aligned} \tag{1.1}$$

where

$$Lu(x, t) \equiv \partial_t u - \sum_{i,j=1}^{n} \partial_{x_i}(a_{ij}\partial_{x_j} u + a_i u) + \sum_{i=1}^{n} b_i \partial_{x_i} u + au \tag{1.2}$$

* This research was supported in part by NSF Grant Number MCS-7801687.
† This reserch was supported in part by NSF Grant Number ENG-7803227.

0022-0396/81/120438–09$02.00/0

and

$$f(x,t) \equiv -g(x,t) + \sum_{i=1}^{n} \partial_{x_i} g_i. \tag{1.3}$$

We suppose that the coefficients a_{ij}, a_i, b_i, a in (1.2) and the data functions g and g_i in (1.3) exhibit the following dependence:

$$\begin{aligned}
a_{ij} &= a_{ij}(x, t; u(y_1, \psi_1(t, u(x,t))),..., u(y_k, \psi_k(t, u(x,t)))),\\
a_i &= a_i(x, t; u(y_1, \psi_1(t, u(x,t))),..., u(y_k, \psi_k(t, u(x,t)))),\\
b_i &= b_i(x, t; u(y_1, \psi_1(t, u(x,t))),..., u(y_k, \psi_k(t, u(x,t)))),\\
a &= a(x, t; u(y_1, \psi_1(t, u(x,t))),..., u(y_k, \psi_k(t, u(x,t)))),\\
g &= g(x, t; u(y_1, \psi_1(t, u(x,t))),..., u(y_k, \psi_k(t, u(x,t)))),\\
g_i &= g(x, t; u(y_1(t, u(x,t))),..., u(y_k, \psi_k(t, u(x,t)))),
\end{aligned} \tag{1.4}$$

where the functions $\psi_i(t, \lambda)$, $i = 1, 2,..., k$ are defined and continuous on $[0, T] \times R^1$ and satisfy

$$0 \leqslant \psi_i(t, u(x,t)) \leqslant t, \qquad i = 1, 2,..., k. \tag{1.5}$$

The parameters y_i, $i = 1, 2,..., k$, denote fixed points in Ω.

We are going to suppose here that the coefficients a_{ij} satisfy the additional condition

$$\nu \sum_{i=1}^{n} \xi_i^2 \leqslant \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \leqslant \mu \sum_{i=1}^{n} \xi_i^2 \tag{1.6}$$

for some positive constants ν, μ.

We will also assume, for the sake of simplicity, that the coefficients and the data appearing in (1.4) are each bounded measurable functions of their arguments and that, in addition, they are each continuous functions of their last k arguments.

In Section 2 of this paper we present a weak formulation of problem (1.1) and state an existence theorem for weak solutions. In Section 3 we define a family of approximate weak solutions by retarding the time variable in the $u(x, t)$ appearing in the arguments of each of the functions ψ_i, $1 \leqslant i \leqslant k$. Compactness of this family is demonstrated and used in Section 4 to complete a proof of the existence theorem.

In order to motivate the problem (1.1) subject to (1.2) through (1.6) we consider the problem of determining a pair of unknown functions $u(x, t)$, $a(\lambda)$ which satisfy

$$\begin{aligned}
&\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(a(u)\frac{\partial u}{\partial x}\right), && x>0,\quad t>0,\\
&u(x,0)=0, && x>0,\\
&u(0,t)=f(t), && t>0,\\
&a(f(t))\frac{\partial u}{\partial x}(0,t)=g(t), && t>0,
\end{aligned}\tag{1.7}$$

where f, g denote known functions. In particular, we must suppose that $f(0)=0$ and $f(t)$ is strictly increasing. If we denote f^{-1} by τ, then (1.7) can be reformulated as follows:

$$\begin{aligned}
&\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(\frac{g(\tau(u(x,t)))}{\partial u(0,\tau(u(x,t)))/dx}\frac{\partial u}{\partial x}\right), && x>0,\quad t>0,\\
&u(x,0)=0, && x>0,\\
&u(0,t)=f(t), && t>0.
\end{aligned}\tag{1.8}$$

If we approximate $\partial u(0,\tau)/\partial x$ by a difference formula, we obtain a problem of the form (1.1) where the coefficient displays a dependence of the form indicated in (1.4).

2. A Weak Formulation of the Problem

Let $L_2(Q_T)$ denote the real Hilbert space of functions which are defined and square integrable in Q_T. In addition, let $W_2^{1,0}(Q_T)$ denote the subspace of $L_2(Q_T)$ consisting of those functions u in $L_2(Q)$ all of whose derivatives $\partial u/\partial x_k$, $k=1,2,\ldots,n$, are in $L_2(Q_T)$. Finally, let $W_2^{1,1}(Q_T)$ denote the subspace of $W_2^{1,0}(Q_T)$ consisting of those functions u in $W_2^{1,0}(Q_T)$ whose time derivative $\partial u/\partial t$ is also in $L_2(Q_T)$. For each of these spaces we have the natural inner product defined as follows:

$$(u,v)_{L_2(Q_T)}=\int_{Q_T} u(x,t)\,v(x,t)\,dx\,dt,\tag{2.1}$$

$$(u,v)_{W_2^{1,0}(Q_T)}=(u,v)_{L_2(Q_T)}+\sum_{k=1}^{n}\left(\frac{\partial u}{\partial x_k},\frac{\partial u}{\partial x_k}\right)_{L_2(Q_T)}\tag{2.2}$$

and

$$(u,v)_{W_2^{1,1}(Q_T)}=(u,v)_{W_2^{1,0}(Q_T)}+\left(\frac{\partial u}{\partial t},\frac{\partial v}{\partial t}\right)_{L_2(Q_T)}.\tag{2.3}$$

Now, following Ladyzenskaya, Sollonikov and Ural'ceva, we let $V_2(Q_T)$ denote the Banach space composed of those elements of $W_2^{1,0}(Q_T)$ satisfying $|u|_{Q_T} < \infty$, where

$$|u|_{Q_T} = \operatorname*{ess\,sup}_{0 \leqslant t \leqslant T} \|u(\cdot, t)\|_{2,\Omega} + \|\nabla u\|_{2,Q_T} \tag{2.4}$$

and

$$\|u(\cdot, t)\|_{2,\Omega}^2 = \int_\Omega u(x,t)^2\, dx, \tag{2.5}$$

$$\|\nabla u\|_{2,Q_T}^2 = \sum_{k=1}^n \left(\frac{\partial u}{\partial x_k}, \frac{\partial u}{\partial x_k}\right)_{L^2(Q_T)}. \tag{2.6}$$

Finally, we let $\mathring{W}_2^{1,0}(Q_T)$, $\mathring{W}_2^{1,1}(Q_T)$ and $\mathring{V}_2(Q_T)$ denote respective subspaces whose elements "vanish on S_T."

Proceeding formally, we multiply Eq. (1.1) by an η in $\mathring{W}_2^{1,1}(Q_T)$ and integrate over Q_T. For η such that $\eta(x, T) = 0$, integrating by parts leads to

$$L(u, \eta) = \int_\Omega \phi(x)\, \eta(x, 0)\, dx, \tag{2.7}$$

where

$$\begin{aligned} L(u, \eta) = &- \left(u, \frac{\partial \eta}{\partial t}\right)_{L_2(Q_T)} + \sum_{i,j=1}^n \left(a_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial \eta}{\partial x_i}\right)_{L_2(Q_T)} \\ &+ \sum_{i=1}^n \left(a_i u + g_i, \frac{\partial \eta}{\partial x_i}\right)_{L_2(Q_T)} \\ &+ \sum_{i=1}^n \left(b_i \frac{\partial u}{\partial x_i}, \eta\right)_{L_2(Q_T)} + (au + g, \eta)_{L_2(Q_T)}. \end{aligned} \tag{2.8}$$

Letting $C(Q_T)$ denote the space of functions defined and continuous on Q_T, the requirement that $u \in \mathring{W}_2^{1,0}(Q_T) \cap C(Q_T)$ is consistent with the existence of all integrals in (2.8) and with the assumptions that the coefficients and data in (1.4) are bounded, measurable functions of all their arguments and are continuous in their last k arguments (see [2, p. 204]). Thus we have the following.

DEFINITION. A weak solution of (1.1) subject to (1.2) through (1.6) is any $u \in \mathring{W}_2^{1,0}(Q_T) \cap C(Q_T)$ such that (2.7) holds for all $\eta \in \mathring{W}_2^{1,1}(Q_T)$ satisfying $\eta(x, T) = 0$ for $x \in \Omega$.

We a now state our main result.

THEOREM. *Suppose that the coefficients and data functions appearing in* (1.4) *are bounded measurable functions on* $Q_T \times R^k$ *and that they are continuous functions of their last k arguments. Suppose also that* $\phi \in L_2(\Omega)$. *Then there exists a weak solution to* (1.1) *subject to* (1.2) *through* (1.6).

3. A FAMILY OF APPROXIMATE SOLUTIONS

For real θ satisfying $0 < \theta < 1$, let K^θ denote a function which satisfies

$$\begin{aligned}
&\text{(i)} \quad K^\theta \in C_c^\infty(R^n),\\
&\text{(ii)} \quad K^\theta \geqslant 0 \qquad \text{and} \qquad K^\theta(x) = 0 \qquad \text{for } |x| > \theta, \qquad (3.1)\\
&\text{(iii)} \quad \int_{R_n} K^\theta(x)\,dx = 1.
\end{aligned}$$

Then define

$$K^\theta * \phi(y_i) = \int_{R^n} \phi(y_i - y)\,K^\theta(y)\,dy, \qquad i = 1,..., k, \tag{3.2}$$

$$K^\theta * u(\xi, \tau) = \int_{R^n} u(\xi - y, \tau)\,K^\theta(y)\,dy, \qquad (\xi, \tau) \in Q_T, \tag{3.3}$$

where we agree to define ϕ and u as zero for all x not in Ω.

We now propose to modify the coefficients a_{ij}, a_i, b_i, and the data functions g_i, g in the following way. Define functions a_{ij} by

$$\begin{aligned}
a_{ij}^\theta = {}& a_{ij}(x, t; K^\theta * \phi(y_1),..., K^\theta * \phi(y_k)), \qquad 0 \leqslant t \leqslant \theta\\
& a_{ij}(x, t; K^\theta * u\, y_1, \psi_1(t-\theta, K^\theta * u(x, t-\theta))),...,\\
&\qquad K^\theta * u(y_k, \psi_k(t-\theta, K^\theta * u(x, t-\theta)))), \qquad 0 \leqslant t \leqslant T. \qquad (3.4)
\end{aligned}$$

Similar definitions are made for a_i^θ, b_i^θ, a^θ as well as for g_i^θ and g^θ. Using these modified coefficients in place of their unretarded counterparts in (2.8) leads to a modified functional which we denote by $L^\theta(u, \eta)$. Then for each θ, $0 < \theta < 1$, we define the approximation u^θ on Q_T to be the unique solution in $V_2(Q_T)$ of the following variational problem: Find $u^\theta \in \dot{V}_2(Q_T)$ satisfying,

$$L^\theta(u^\theta, \eta) = \int_\Omega \phi(x)\,\eta(x, 0)\,dx, \tag{3.5}$$

for all $\eta \in \dot{W}_2^{1,1}(Q_T)$ such that $\eta(x, T) = 0$.

LEMMA 1. *For each θ, $0 < \theta < 1$, u^θ is well defined.*

Proof. Evidently we must demonstrate the existence and uniqueness of the solution to the variational problem (3.5). Consider first the variational problem in Q_θ. There the coefficients and the data in the functional L^θ are well defined, and by Theorem 4.1 in Ladyzhenskaya, Sollonikov and Ural'ceva [2, p. 153], u^θ exists and is unique in Q_θ. Moreover, from Theorem 2.1 [2, p. 143] we have

$$|u^\theta|_{Q_\theta} \leqslant C_1$$

for some constant C_1 depending only on η, ν, μ, the bounds on the a_{ij}^θ, a_i^θ, b_i^θ, a^θ, g_i^θ, g^θ, the measure of Ω, T, and on $\|\phi\|_{L_2(\Omega)}$. In addition, Theorem 8.1 [2, p. 192] implies that for $\delta > 0$ and for $Q_{\delta,\theta} = \Omega \times (\delta, \theta]$ we have

$$\operatorname*{ess\,sup}_{Q_{\delta,\theta}} |u^\theta| < C_2$$

for some constant C_2 depending only on C_1, ν, μ, the measure of Ω, T, δ, and on the bounds on the data a_{ij}^θ, a_i^θ, b_i^θ, a^θ, g_i^θ, g^θ. Finally, an application of Theorem 10.1 [2, p. 204] yields the Hölder continuity of u^θ in $Q_{2\delta,\theta}$ with exponent α depending on n, ν, μ, the measure of Ω, the smoothness of $\partial\Omega$, T, and on the data bounds. The Hölder norm $|u^\theta|_{Q_{2\delta,\theta}}^{(\alpha)}$ is bounded by a constant C_3 which depends on n, C_2, ν, μ, the measure of Ω, T, δ, and on the data bounds.

Next we consider the variational problem in $Q_{\theta,2\theta}$ with initial data $u^\theta(x, \theta)$. Since u^θ is well defined in Q_θ, the coefficients and data in L^θ are well defined in $Q_{\theta,2\theta}$. Consequently we can extend the solution into $Q_{\theta,2\theta}$, and, by induction, to all of Q_T. Repeating the application of the above theorems, we obtain

$$|u^\theta|_{Q_T} \leqslant C_1, \tag{3.6}$$

$$\operatorname*{ess\,sup}_{Q_{\delta,T}} |u^\theta| \leqslant C_2, \tag{3.7}$$

and

$$|u^\theta|_{Q_{\delta,T}}^{(\alpha)} \leqslant C_3, \tag{3.8}$$

where the constants C_1, C_2, C_3 and the exponent α depend on the various ingredients of the problem just as indicated in the above discussion. This proves the lemma.

We conclude this section with the following compactness result.

LEMMA 2. *The family u^θ, $0 < \theta < 1$, is weakly compact in $L_2(Q_T)$ and in $W_2^{1,0}(Q_T)$. Moreover, the family u^θ, $0 < \theta < 1$, is uniformly bounded and equicontinuous in $Q_{\delta,T}$ for each $\delta > 0$.*

Proof. It follows from (3.1)-(3.4) that the bounds for a_{ij}^θ, a_i^θ, b_i^θ, a^θ, g_i^θ, g^θ are identical to the respective bounds for a_{ij}, a_i, b_i, a, g_i, g. Then the constants C_i, $i = 1, 2, 3$, and α in (3.6)-(3.8) do not depend on θ and the lemma follows.

4. PROOF OF THE THEOREM

Let θ_n denote a sequence of positive numbers decreasing monotonically to zero and let u^{θ_n} denote the corresponding sequence of solutions to (3.5). Lemma 2 implies the existence of a subsequence u^{θ_m} of solutions converging weakly in $L_2(Q_T)$ and in $W_2^{1,0}(Q_T)$ to a limit u. Lemma 2 implies further than the subsequence can be selected so that for each $\delta > 0$ the convergence is uniform on $Q_{\delta,T}$. Then a diagonalization argument implies that $u \in C(Q_T)$. Since $u^{\theta_m} \in \dot{V}_2(Q_T)$ $\forall m$ it is clear that $u \in \dot{V}_2(Q_T)$ as well.

Consider now for $m = 1, 2, \ldots$ the equations

$$L^{\theta_m}(u^{\theta_m}, \eta) = \int_\Omega \phi(x)\, \eta(x, 0)\, dx. \tag{4.1}$$

By adding and subtracting $L(u, \eta)$ on the left side of (4.1) we obtain

$$L(u, \eta) + \varepsilon_m = \int_\Omega \phi(x)\, \eta(x, 0)\, dx, \qquad m = 1, 2, \ldots, \tag{4.2}$$

where

$$\varepsilon_m = L^{\theta_m}(u^{\theta_m}, \eta) - L(u, \eta), \qquad m = 1, 2, \ldots. \tag{4.3}$$

Then our proof of the theorem will be complete if we can show that

$$\lim_{m\to\infty} \varepsilon_m = 0. \tag{4.4}$$

In proving (4.4) it will suffice to consider only the terms involving $a_{ij}^{\theta_m}$ and a_{ij} since the remaining terms are then handled in a similar manner.

:t

$$I_m = \int_{Q_T} a_{ij}^{\theta_m} \frac{\partial u^{\theta_m}}{\partial x_j} \frac{\partial \eta}{\partial x_i}\, dx\, dt - \int_{Q_T} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial n}{\partial x_i}\, dx\, dt$$

$$= \int_{Q_T} \{a_{ij}^{\theta_m} - a_{ij}\} \frac{\partial u^{\theta_m}}{\partial x_j} \frac{\partial \eta}{\partial x_i}\, dx\, dt$$

$$+ \int_{Q_T} a_{ij} \left\{\frac{\partial u^{\theta_m}}{\partial x_j} - \frac{\partial u}{\partial x_j}\right\} \frac{\partial \eta}{\partial x_j}\, dx\, dt,$$

$$I_m = J_m^1 + J_m^2. \tag{4.5}$$

Clearly J_m^2 must tend to zero as m tends to infinity in view of the weak convergence of u^{θ_m} to u in $W_2^{1,0}(Q_T)$. Consider then the term J_m^1 in (4.5).

Suppose that

$$\lim_{m\to\infty} a_{ij}^{\theta_m} = a_{ij} \qquad \text{almost everywhere in } Q_T. \tag{4.6}$$

Then it follows from Egorov's [1, p. 42] theorem that for any $\varepsilon > 0$ there is a subset Q_T' of Q_T such that the measure of $Q_T - Q_T'$ is less than ε and $a_{ij}^{\theta_m}$ converges uniformly to a_{ij} on Q_T'.

Now write

$$J_m^1 = \int_{Q_\delta} \{a_{ij}^{\theta_m} - a_{ij}\} \frac{\partial u^{\theta_m}}{\partial x_j} \frac{\partial \eta}{\partial x_i}\, dx\, dt$$

$$+ \int_{Q'_{\delta,T}} \{a_{ij}^{\theta_m} - a_{ij}\} \frac{\partial u^{\theta_m}}{\partial x_j} \frac{\partial \eta}{\partial x_i}\, dx\, dt$$

$$+ \int_{W_{\delta,T}} \{a_{ij}^{\theta_m} - a_{ij}\} \frac{\partial u^{\theta_m}}{\partial x_j} \frac{\partial \eta}{\partial x_i}\, dx\, dt, \tag{4.7}$$

where, for $\delta > 0$, $Q'_{\delta,T} = Q_{\delta,T} \cap Q_T'$ and $W_{\delta,T} = Q_{\delta,T} - Q'_{\delta,T}$. If we suppose that

$$\operatorname*{ess\,sup}_{Q_T} |a_{ij}| < C_4, \tag{4.8}$$

then we have

$$|J_m^1| \leqslant 2C_4 |u^{\theta_m}|_{Q_T} \left\| \frac{\partial \eta}{\partial x_i} \right\|_{L_2(Q_\delta)} + 2C_4 |u^{\theta_m}|_{Q_T} \left\| \frac{\partial \eta}{\partial x_i} \right\|_{L_2(W_{\delta,T})}$$

$$+ \operatorname*{ess\,sup}_{Q'_{\delta,T}} |a_{ij}^{\theta_m} - a_{ij}|\, |u^{\theta_m}|_{Q_T} \left\| \frac{\partial \eta}{\partial x_i} \right\|_{L_2(Q_T)}.$$

Using (3.6) and the fact that $\partial\eta/\partial x_i \in L_2(Q_T)$ it is evident that the first term on the right side of (4.9) can be made arbitrarily small by choosing $\delta > 0$ sufficiently small. Similarly, the second term on the right side of (4.9) can be made small by choosing $\varepsilon > 0$ small. Finally, the third term on the right side of that expression can be made as small as we like by choosing m sufficiently large. We conclude that $\lim_{m\to\infty} J_m^1 = 0$, provided that (4.6) holds.

To prove that (4.6) must hold, it will be sufficient to show that

$$\forall \delta > 0 \quad a_{ij}^{\theta_m} \to a_{ij} \qquad \text{pointwise in } Q_{\delta,T}, \tag{4.10}$$

since in this case the $a_{ij}^{\theta_m}$ converge to a_{ij} in measure on Q_T which in turn implies the existence of a subsequence which converges a.e. on Q_T.

To prove (4.10) note that for $\delta > 0$, u^{θ_m} tends uniformly to u on $Q_{\delta,T}$ and hence

$$\lim_{m\to\infty} K^{\theta_m} * u^{\theta_m}(x, t-\theta_m) = u(x, t) \tag{4.11}$$

when the convergence is uniform. It follows that for $l = 1, 2,..., k$,

$$\lim_{m\to\infty} \psi_l(t-\theta_m, K^{\theta_m} * u^{\theta_m}(x, t-\theta_m)) = \psi_l(t, u(x, t)). \tag{4.12}$$

and

$$\lim_{m\to\infty} K^{\theta_m} * u^{\theta_m}(y_l, \psi_l((t-\theta_m), K^{\theta_m} * u^{\theta_m}(x, t-\theta_m)))$$

$$= u(y_l, \psi_l(t, u(x, t)))$$

where we have used the continuity of the ψ_l. Finally, from the continuity of the a_{ij} in the last k variables follows the pointwise convergence of $a_{ij}^{\theta_m}$ to a_{ij} on $Q_{\delta,T}$ and this concludes the proof of the theorem.

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